REAL-LIFE APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT:

This study introduces real-life mathematical models of international relationships suitable for ordinary differential equations, by investigating conflicts between different nations or alliances. The system of differential equations are constructedbased on the work of Richardson. The solutions and the stability of systems of Oridinary Differential Equations are observed.

KEY WORDS: ODE, Arms race, Eigen values, Critical point, Armaments.

INTRODUCTION:

Ordinary differential equations (ODEs), especially systems of ODEs, have been applied in many fields such as physics, electronic engineering and population dynamics. This is a powerful tool for analysing the relationship between various dynamic quantities.In this paper we focus on the study of models describing international conflicts, which were originally derived by Lewis Fry Richardson, describing the relationship between two nations or two alliances that deem war to be imminent. In particular, he devised mathematical models of arms races using differential equations. One assumes that if one country increases its weapons, another country will do the same. Sequentially, the first country responds by storing more weapons. Richardson proposed that this kind of arms race can be represented by a pair of differential equations. Richardson's model of international relations, which includes an arm race, used for discussing stability, is analogous to the differential equations in the predator-prey model. In the present paper, firstly, a simpler model of an arms race is depicted. More realistic models are constructed, and additional factors that influence the relationship between two nations or alliances, such as the cost of armaments, the grievances between nations and their ambitions, are considered.

1. SIMPLE ARMS RACE FOR DIFFERENTIAL EQUATIONS:

It is a well-known fact that an increase in armaments is one of the primary reasons for war. Another reason is the unsolvable conflict of ambitions, such as occupying more territory or recovering tracts of land.

We assume that if one nation increases its armaments, then the opposing nation will do likewise because it assumes that the balance of power will be negatively affected.

Let x(t) be the armaments of nation X, and y(t) be the armaments of nation Y at time t. The rate of change of the armaments on one side depends on the number of armaments on the opposing side, because if one nation increases its armaments, the other will follow suit. That is, dx/dt (or dy/dt) is proportional to y (or x).We assign constants of proportionally h and 1 to x and y, respectively, which represent the efficiency of increasing armaments.

Hence, we can establish a system of differential equations in the following form:

$$dx/(dt)=kx \qquad (1.1)$$
$$dy/dt=ky$$

Thissystemcanbeusedtodescribetherelationshi pbetweentwonationsoralliances,eachofwhichdecidestodefenditselfagainstpossibleattackbytheother.

It is easy to obtain the solutions for the system (1.1) which we give as follows:

$x(t)=\sqrt{(k/l)}\ (Ae^{t\sqrt{kl}} - Be^{-t\sqrt{kl}})$,

$y(t)=(Ae^{t\sqrt{kl}} - Be^{-t\sqrt{kl}})$ (1.2)

Given initial conditions,

$x(0)=x_0, y(0)=y_0$

we can obtain

$$A = \frac{1}{2}\left(y_0+\sqrt{l/k}\, x_0\right) \quad (1.3)$$

$$B = \frac{1}{2}\left(y_0-\sqrt{l/k}\, x_0\right)$$

It is possible to estimate the values of k and l. For example, when y remains a constant C, it follows from (1.1)that

(1.4) $\frac{1}{k} = C/(dx/dt) = C\ dt/dx$

Solving (1.4) we obtain

(1.5) $\frac{1}{k} x = Ct + b$

Assuming x(0) = 0, it follows from (2.5)

that b = 0and

$\frac{1}{k} = \frac{C}{x} t$ for $x>0$

Hence when Xhas caught up to Y, which means X= C, we have$1/k=t$ Thus $1/k$ is the time required for nation Xto catch up with the armaments of Y provided that y remains constant. Richardson also observed that k is proportional to the amount of industry in a country.

2. CONSTRUCTION OF A REALISTIC MODEL

The relationship between nations or alliances in the real world is more complicated. We therefore need to modify the system (2.1) by considering more factors thataffectthechangeratesdx/dtanddy/dtinanefforttoadaptittotherealworld. Richardsonconstructeddifferentialequationsofconflict,takingintoaccountfactors suchasthecostofarmaments,grievancesorambitionsbetweennations,etc([1], 1993;[11],1957).Thesystemconstructedfordescribingtherelationshipbetween the nations or alliances, X and Y, is asfollows:

$dx/dt=ky-\alpha x+g$ (2.1)

$dy/dt=lx-\beta y+h,\quad k,l,\alpha,\beta,g,h>0$

where x(t) (respectively y(t)) denotes the armaments of nation X(Y); k(l) is the efficiency of increasing the armaments of X(Y); g(h) is the ambitions of or grievances that X(Y) has towards Y(X) , affecting dx/dt (dy/dt) positively the influence of the cost of armaments is a restraining factor, represented by $-\alpha x(\beta y)$.

We encourage students to observe the stability of system (1.1) by finding and analysing the critical point. This is an excellent exercise for students to familiarise themselves with the classification of a critical point by stability and type, utilising the eigenvalues. The stability of the critical point of the system (2.1) depends on the logical relationship between kl and αβ.

It would entail a great deal of work to simplify the solutions to system (2.1).

Given the initial conditions $x(0) = x_0, y(0) = y_0$, the unique solution to (2.1) can be written as follows:

$$x(t)=x_1+\frac{kl}{2\mu(\mu+\omega)} Ae^{\lambda_1 t}+\frac{1}{2}\frac{(\mu+\omega)}{\mu} Be^{\lambda_2 t} \quad (2.2)$$

$$y(t)= y_1+\frac{l}{2\mu} Ae^{\lambda_1 t}-\frac{l}{2\mu} Be^{\lambda_2 t} \quad (2.3)$$

Where$(x_1,y_1)=\left(\frac{\beta g+hk}{\alpha\beta-kl},\frac{\alpha h+gl}{\alpha\beta-kl}\right)$

Is the critical point

$$\lambda_1=-\frac{1}{2}(\alpha+\beta)+\frac{1}{2}\sqrt{(\alpha-\beta)^2+4kl}$$

$$\lambda_2=-\frac{1}{2}(\alpha+\beta)-\frac{1}{2}\sqrt{(\alpha-\beta)^2+4kl}$$

Are Eigen values and

$\omega = 1/2 \ (\alpha - \beta), \ \mu = \sqrt{}(\omega$ ^2+kl) (2.4)

A=(x_0+g/λ_1)+(μ+ω)/l (y_0+h/λ_1) (2.5)

B=(x_0+g/λ_2)-k/(μ+ω) (y_0+h/λ_2) (2.6)

Finally, we estimate the coefficients in our model. It is interesting to note that Richardson estimates α^(-1) (β^(-1)) to be thelifetime of X's (Y's) parliament ([2]). For example, since the lifetime of Britain's parliament is five years, we obtain α= 0.2 for that country.

To estimate k and l , for example, we consider g = 0 and y = C. Hence

dx/dt=kc-αx

Assuming x(0) = 0 and solving the above equation, we obtain:

x= kC/α (1-e^(-αt)) (2.7)

Substituting the power series expansion

e^(-αt)=∑_(n=0)^ ∞ ▒((□ -αt) □ ^n)/n!=1-αt+⋯……….

In (2.7), since αt<1 ,we have

x≈ kC/α αt for t>0

That is 1/k = C/x t

Thus 1/k isthetimerequiredforX tocatchuptoY,ifY'sarmamentsonlyremain a constant C. We recall that k represents the product efficiency of armaments of nationX.Itisobviousthatausefulexerciseistoapply theknowledgetoseries.

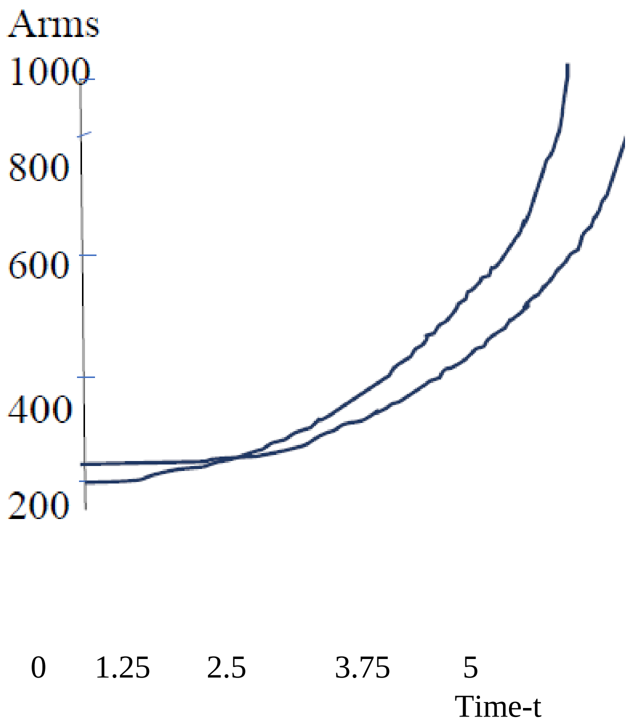

FIGURE1:Solutionstothemodel(3.1)with k=O.6,l=O.8,α=β=O.2,x_0= 100, y_0=8O.

expansion of functions. We note that our method of estimating k is different from the method given in [2], in which the constant k is estimated in the following way, assuming g=0 and y =y_1:

when x = 0,

1/k=y_1/(dx/dt).

A future work could include the collection of data and information on historical wars to construct mathematical models. For example, during the Cold War, both the USA and the Soviet Union were involved in an arms race for conventional and nuclear weapons.

Here we only give some examples and graphs, choosing different values for the constants k,l,α,β. We assume g = h = 0, so x_1=y_1=0. We use a solid line to represent the solution x(t), and a grey line for the solution y(t).

If k = 0.6, l = 0.8, α=β=0.2, x_0=100, y_0=80, the solutions and graphs are as follows (unstable case, αβ<kl,λ_1>0>λ_2) (see Figure 1)

x(t)=84.641 exp (0.49282t)+15.359 exp (—0.89282t) y(t)=97.735 exp (049288t)-17.735 exp (—0.89282t)

If k = 0.35, l = 0.4, α=0.3,β=0.5, x_0=100, y_0=95,

the solutions and graphs are as follows (unstable case, αβ<kl,λ_2<λ_1<0) (see Figure 1)

x(t)=105.84 exp (-1.2702*□10□^(-2) t)-5.8355 exp (—0.7873t)

y(t)= 8.1247 exp (-0.7873t) +86.875 exp (-1.2702*□10□^(-2) t)

Conclusion:

We believe that more research could be conducted into the mathematical modelling of international relationships at undergraduate level. Such modelling also has realistic applications in military, business and other fields. From an educational perspective, these mathematical models are also realistic applications of ordinary differential equations (ODEs) — hence the proposal that these models should be added to ODE textbooks as flexible and vivid examples to illustrate and study differential equations.

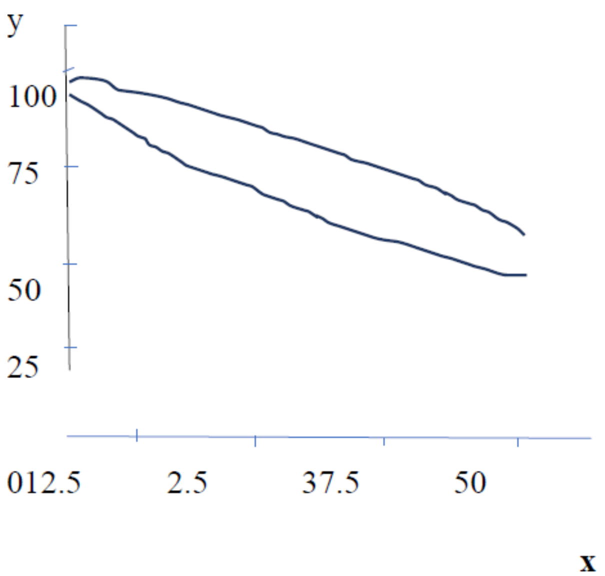

Figure2: Solutions to the model (3.1) with k=0.6,l=0.8 α=β=0.2,x_0=100,y_0=80.

REFERENCES:

[1] ASHFORD, O.M. 1993. Collected papers of Lewis Fry Richardson. Cambridge.

[2]DENNIS,G.Z.&MICHAEL,R.C.1997.Differential equationswithboundarysa1uepvob1ems. Brooks &Cole.

[3]JOUBERT,S.V.2002.Technicon MathematicsIIIwithDERIVE5.GreatWhitePublishers,Faerie Glen,Ganteng,SouthAfrica.

[4]KREYSZIG,E.1999.AdvancedEngineering mathematics.8thed.NewYork:Wiley&Sons.

[5]LANCHESTER, F.W. 1996. Aircraft in warfare: the down of the fourth arm. London: Constable.

[6]RICHARDSON,L.F. 1960a.Armsandinsecurity:amathematicalstudyoftheCAUSESandoriginsofwar. EditedbyN.Rashesky&E.Trucco.Pittsburgh:BoxwoodPress..